Formule iz Diferencijalne geometrije

Duljina luka krivulje: $\alpha \ldots \ \vec{r} = \vec{r}(t)$ $\quad s = \int_{t_1}^{t_2} \left|\dot{\vec{r}}(t)\right| dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

Frenetov trobrid

$\alpha \ldots \ \vec{r} = \vec{r}(s)$ $\quad \vec{t^o}(s) = \frac{d\vec{r}}{ds} = \vec{r}'(s); \quad \vec{n^o}(s) = \frac{\vec{t^o}'}{|\vec{t^o}'|} = \frac{\vec{r}''(s)}{|\vec{r}''(s)|}; \quad \vec{b^o}(s) = \vec{t^o}(s) \times \vec{n^o}(s)$

$\alpha \ldots\ldots \vec{r} = \vec{r}(t)$ $\quad \vec{t^o} = \vec{r}' = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}; \quad \vec{n^o} = \frac{(\dot{\vec{r}} \times \ddot{\vec{r}}) \times \dot{\vec{r}}}{|(\dot{\vec{r}} \times \ddot{\vec{r}}) \times \dot{\vec{r}}|}; \quad \vec{b^o} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{|\dot{\vec{r}} \times \ddot{\vec{r}}|}$

Normalna (N), rektifikaciona (R) i oskulaciona (O) ravnina

$\alpha \ldots \ \vec{r} = \vec{r}(s)$, $\quad \alpha \ldots \ \vec{r} = \vec{r}(t)$

$x_o'(x - x_o) + y_o'(y - y_o) + z_o'(z - z_o) = 0$ (N) $\quad \dot{x}_o(x - x_o) + \dot{y}_o(y - y_o) + \dot{z}_o(z - z_o) = 0$

$x_o''(x - x_o) + y_o''(y - y_o) + z_o''(z - z_o) = 0$ (R) $\quad \begin{vmatrix} x - x_o & y - y_o & z - z_o \\ \dot{x}_o & \dot{y}_o & \dot{z}_o \\ l & m & n \end{vmatrix} = 0$

$\left(l = \begin{vmatrix} \dot{y}_o & \dot{z}_o \\ \ddot{y}_o & \ddot{z}_o \end{vmatrix}, \quad m = \begin{vmatrix} \dot{z}_o & \dot{x}_o \\ \ddot{z}_o & \ddot{x}_o \end{vmatrix}, \quad n = \begin{vmatrix} \dot{x}_o & \dot{y}_o \\ \ddot{x}_o & \ddot{y}_o \end{vmatrix} \right)$

$\begin{vmatrix} x - x_o & y - y_o & z - z_o \\ x_o' & y_o' & z_o' \\ x_o'' & y_o'' & z_o'' \end{vmatrix} = 0$ (O) $\quad \begin{vmatrix} x - x_o & y - y_o & z - z_o \\ \dot{x}_o & \dot{y}_o & \dot{z}_o \\ \ddot{x}_o & \ddot{y}_o & \ddot{z}_o \end{vmatrix} = 0$

Fleksija i torzija

$\alpha \ldots \ \vec{r} = \vec{r}(s)$ Fleksija: $\kappa(s) = \left|\frac{d\vec{t}}{ds}\right| = |\vec{r}''(s)|$ $\quad$ Torzija: $\tau(s) = -\vec{n}^o \cdot \vec{b}^{o\prime} = \frac{(\vec{r}', \vec{r}'', \vec{r}''')}{|\vec{r}''|^2}$

$\alpha \ldots \ \vec{r} = \vec{r}(t)$ $\quad$ Fleksija: $\kappa(t) = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$ $\quad$ Torzija: $\tau(t) = \frac{(\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}})}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$

Prva diferencijalna forma plohe $\vec{r} = \vec{r}(u, v)$

$$\mathrm{I} \equiv ds^2 = \left(d\vec{r}\right)^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2; \qquad E = \vec{r_u} \cdot \vec{r_u}, \quad F = \vec{r_u} \cdot \vec{r_v}, \quad G = \vec{r_v} \cdot \vec{r_v}$$

Duljina luka krivulje $\vec{r}(t) = \vec{r}(u(t), v(t))$ na plohi $\vec{r} = \vec{r}(u, v)$

$$s = \int_{t_1}^{t_2} \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\, dt$$

Kut između dviju krivulja na plohi $\vec{r} = \vec{r}(u, v)$

$$\cos\omega = \left(E\,du\,d\bar{u} + F(du\,d\bar{v} + d\bar{u}\,dv) + G\,dv\,d\bar{v}\right) \Big/ \left(\sqrt{\left(E\,du^2 + 2F\,du\,dv + G\,dv^2\right)}\sqrt{\left(E\,d\bar{u}^2 + 2F\,d\bar{u}\,d\bar{v} + G\,d\bar{v}^2\right)}\right)$$

Površina područja D **na plohi** $\vec{r}=\vec{r}(u,v)$: $\quad S=\iint_D \left|\vec{r}_u\times\vec{r}_v\right| du\,dv=\iint_D \sqrt{EG-F^2}\,du\,dv$

Druga diferencijalna forma plohe $\vec{r}=\vec{r}(u,v)$: $\quad \mathrm{II}\equiv L\,du^2+2M\,du\,dv+N\,dv^2$

$$L=\vec{N}^{\mathrm{o}}\cdot\vec{r}_{uu}=\frac{1}{W}[(\vec{r}_u\times\vec{r}_v)\cdot\vec{r}_{uu}]=\frac{1}{W}(\vec{r}_u,\vec{r}_v,\vec{r}_{uu})$$

$$M=\vec{N}^{\mathrm{o}}\cdot\vec{r}_{uv}=\frac{1}{W}[(\vec{r}_u\times\vec{r}_v)\cdot\vec{r}_{uv}]=\frac{1}{W}(\vec{r}_u,\vec{r}_v,\vec{r}_{uv})$$

$$N=\vec{N}^{\mathrm{o}}\cdot\vec{r}_{vv}=\frac{1}{W}[(\vec{r}_u\times\vec{r}_v)\cdot\vec{r}_{vv}]=\frac{1}{W}(\vec{r}_u,\vec{r}_v,\vec{r}_{vv})$$

Normalna zakrivljenost plohe $\vec{r}=\vec{r}(u,v)$: $\quad K_n=\dfrac{L\,du^2+2M\,du\,dv+N\,dv^2}{E\,du^2+2F\,du\,dv+G\,dv^2}=\dfrac{\mathrm{II}}{\mathrm{I}}$

Izračunavanje glavnih zakrivljenosti K_1, K_2: $\quad K_n^2-\dfrac{EN-2MF+GL}{EG-F^2}K_n+\dfrac{LN-M^2}{EG-F^2}=0$

Glavni smjerovi μ_1,μ_2:

$$\begin{vmatrix}\mu^2 & -\mu & 1\\ G & F & E\\ N & M & L\end{vmatrix}=0$$

Glavne krivulje zakrivljenosti:

$$\begin{vmatrix}du^2 & -du\,dv & dv^2\\ G & F & E\\ N & M & L\end{vmatrix}=0$$

Eulerova formula: $\quad K_n=K_1\cos^2\alpha+K_2\sin^2\alpha$

Christoffelovi simboli 1. vrste izraženi pomoću Gaussovih veličina 1. reda i njihovih derivacija

$$\vec{r}_{uu}\cdot\vec{r}_u=\begin{bmatrix}u\;u\\u\end{bmatrix}=\frac{1}{2}E_u,\quad \vec{r}_{uu}\cdot\vec{r}_v=\begin{bmatrix}u\;u\\v\end{bmatrix}=F_u-\frac{1}{2}E_v,\quad \vec{r}_{uv}\cdot\vec{r}_u=\begin{bmatrix}u\;v\\u\end{bmatrix}=\frac{1}{2}E_v,$$

$$\vec{r}_{uv}\cdot\vec{r}_v=\begin{bmatrix}u\;v\\v\end{bmatrix}=\frac{1}{2}G_u,\quad \vec{r}_{vv}\cdot\vec{r}_u=\begin{bmatrix}v\;v\\u\end{bmatrix}=F_v-\frac{1}{2}G_u,\quad \vec{r}_{vv}\cdot\vec{r}_v=\begin{bmatrix}v\;v\\v\end{bmatrix}=\frac{1}{2}G_v.$$

Christoffelovi simboli 2. vrste izraženi pomoću Gaussovih veličina 1. reda i njihovih derivacija

$$\begin{Bmatrix}u\;u\\u\end{Bmatrix}=\Gamma^u_{uu}=\frac{GE_u-2FF_u+FE_v}{2(EG-F^2)},\quad \begin{Bmatrix}u\;u\\v\end{Bmatrix}=\Gamma^v_{uu}=\frac{-FE_u+2EF_u-EE_v}{2(EG-F^2)},\quad \begin{Bmatrix}u\;v\\u\end{Bmatrix}=\Gamma^u_{uv}=\frac{GE_v-FG_u}{2(EG-F^2)},$$

$$\begin{Bmatrix}u\;v\\v\end{Bmatrix}=\Gamma^v_{uv}=\frac{EG_u-FE_v}{2(EG-F^2)},\quad \begin{Bmatrix}v\;v\\u\end{Bmatrix}=\Gamma^u_{vv}=\frac{-FG_v+2GF_v-GG_u}{2(EG-F^2)},\quad \begin{Bmatrix}v\;v\\v\end{Bmatrix}=\Gamma^v_{vv}=\frac{EG_v-2FF_v+FG_u}{2(EG-F^2)}.$$

"Theorema egregium":

$$K=-\frac{1}{4W^4}\begin{vmatrix}E & E_u & E_v\\ F & F_u & F_v\\ G & G_u & G_v\end{vmatrix}+\frac{1}{2W}\left[\frac{\partial}{\partial u}\left(\frac{F_v-G_u}{W}\right)+\frac{\partial}{\partial v}\left(\frac{F_u-E_v}{W}\right)\right]$$

Geodetska zakrivljenost: $\quad K_g=\kappa\sin\theta$

Geodetske linije: $\quad K_g=0$

Krivulja α... $\vec{r}=\vec{r}(s)\;\rightarrow\; K_g=\left(\vec{t}^{\,\mathrm{o}},\vec{t}^{\,\mathrm{o}\prime},\vec{N}^{\mathrm{o}}\right)$;

Krivulja α .....$\vec{r}=\vec{r}(t)\;\rightarrow\; K_g=\dfrac{\left(\dot{\vec{r}},\ddot{\vec{r}},\vec{N}^{\mathrm{o}}\right)}{|\dot{\vec{r}}|^3}$

$$\alpha\;....\;\vec{r}=\vec{r}(u(t),\,v(t))\;\rightarrow\;K_g=\frac{W}{(E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2)^{\frac{3}{2}}}\begin{vmatrix}\dot{u} & \ddot{u}+\begin{Bmatrix}u\;u\\u\end{Bmatrix}\dot{u}^2+2\begin{Bmatrix}u\;v\\u\end{Bmatrix}\dot{u}\dot{v}+\begin{Bmatrix}v\;v\\u\end{Bmatrix}\dot{v}^2\\ \dot{v} & \ddot{v}+\begin{Bmatrix}u\;u\\v\end{Bmatrix}\dot{u}^2+2\begin{Bmatrix}u\;v\\v\end{Bmatrix}\dot{u}\dot{v}+\begin{Bmatrix}v\;v\\v\end{Bmatrix}\dot{v}^2\end{vmatrix}$$